Continuity of function of two Variables.

The function $f(x,y)$ is said to be continuous at (a,b) if $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists and is equal to its functional value $f(a,b)$ at (a,b).

ie. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Ex 1. Show that the function $f(x,y) = x^2+y-1$ is continuous at $(1,-2)$.

$$\lim_{x\to 1}\lim_{y\to -2} x^2+y-1 = \lim_{x\to 1} x^2-2-1$$
$$= \lim_{x\to 1} x^2-3$$
$$= 1-3 = -2.$$

$$\lim_{y\to -2}\lim_{x\to 1} x^2+y-1 = \lim_{y\to -2} 1+y-1$$
$$= \lim_{y\to -2} y = -2$$

Hence $\lim\limits_{(x,y)\to(1,-2)} x^2+y-1 = -2$

Also $f(1,-2) = 1^2-2-1 = 1-3 = -2$

$$\Rightarrow \lim_{(x,y)\to(1,-2)} x^2+y-1 = f(1,-2)$$

Hence $f(x,y)$ is continuous at $(1,-2)$.

Discuss the continuity of $f(x,y)$ at $(0,0)$ [Dr. Heena]

where $f(x,y) = \begin{cases} \dfrac{2xy^2}{x^3+y^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

Let $(x,y) \to (0,0)$ along the line $y = mx$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^3+y^3}$$

$$= \lim_{(x,y)\to(0,0)} \frac{2x \cdot m^2x^2}{x^3+m^3x^3}$$

$$= \lim_{x\to 0} \frac{x^3}{x^3} \cdot \frac{2m^2}{(1+m^3)}$$

$$= \frac{2m^2}{1+m^3}$$ which is not unique as it assumes different values for different values of m.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

$\Rightarrow$ f is not continuous at $(0,0)$.

$\boxed{\text{Partial Derivatives}}$

Let $z = f(x,y)$ be a function of two variables x and y. Then partial derivative of z w.r.t independent variable x $\left(\frac{\partial z}{\partial x}\right)$ is the ordinary derivative of z when y is regarded as a constant.

Function of Two Variables

Similarly partial derivative of z w.r.t. $y\left(\frac{\partial z}{\partial y}\right)$ is the ordinary derivative of z when x is regarded as a constant.

Partial Derivatives of first order

Let $z = f(x,y)$ be a real valued function of two real independent variable, then

$$\lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$, if it exists, is called partial derivative of f w.r.t. x.

It is denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$

Similarly $$\lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$, if it exists, is called partial derivative of f w.r.t. y

It is denoted by $\frac{\partial f}{\partial y}$ or $\frac{\partial z}{\partial y}$

Geometrical Interpretation of Partial Derivative

$z = f(x,y)$ represents a surface.

Now $z = f(x,b)$ represent a curve which is intersection of the surface $z = f(x,y)$ and $y = b$.

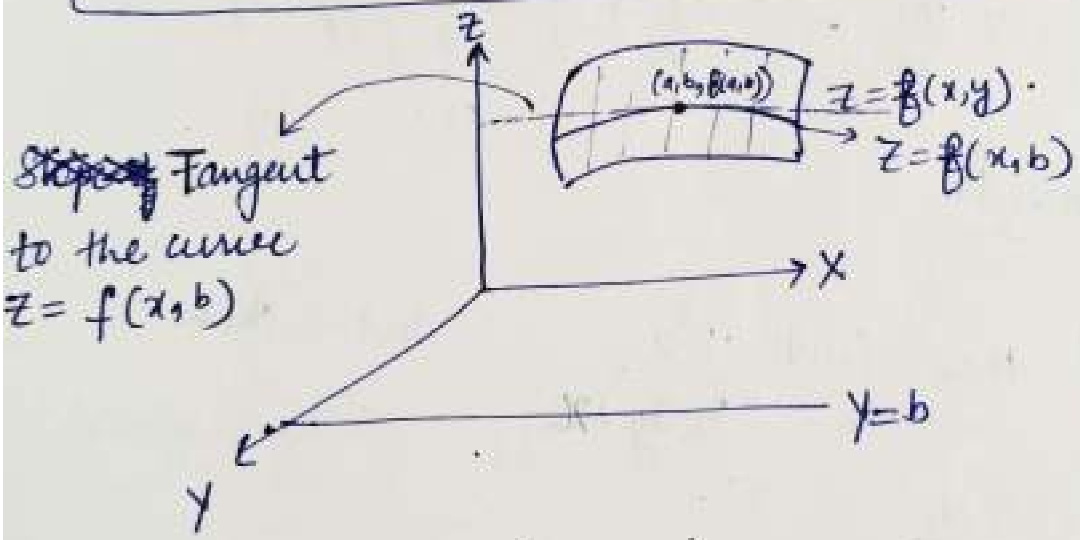
Now $z = f(x,b)$ is a function of one variable and $\frac{d}{dx}[f(x,b)]$ at $x = a$ represents the slope of tangent to the curve $z = f(x,b)$ at point $(a, f(a,b))$.

$\frac{d}{dx}[f(x,b)]$ at $x=a$ is $f_x(a,b)$

i.e. it represents the partial derivative of $f(x,y)$ at (a,b).

$\therefore$ partial derivative f_x at (a,b) represents the slope of the tangent to the curve $z=f(x,y)$ and $y=b$.

partial derivative f_y at (a,b) represents the slope of the tangent to the curve $z=f(x,y)$ and $x=a$.



Ex 1. Let $u(x,y,z) = x^3+y^3+z^3+3xyz$

Find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$.

$\frac{\partial u}{\partial x} = 3x^2+3yz$ $\qquad$ $\frac{\partial u}{\partial y} = 3y^2+3xz$

$\frac{\partial u}{\partial z} = 3z^2+3xy$.

Ex 2. Let $u = x^2(y-z) + y^2(z-x) + z^2(x-y)$.

Find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ (Do. urself).

Ex. 3. If $u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}$ show that

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + z \cdot \frac{\partial u}{\partial z} = 0.$$

$$\frac{\partial u}{\partial x} = 0 - \frac{z}{x^2} + \frac{1}{y}.$$

$$\frac{\partial u}{\partial y} = \frac{1}{z} + 0 - \frac{x}{y^2}.$$

$$\frac{\partial u}{\partial z} = \frac{-y}{z^2} + \frac{1}{x} + 0.$$

Now

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + z \cdot \frac{\partial u}{\partial z} = -\frac{z}{x} + \cancel{\frac{x}{y}} + \cancel{\frac{y}{z}} - \cancel{\frac{x}{y}} - \cancel{\frac{y}{z}} + \frac{z}{x}$$

$$= 0.$$

Hence proved.

Ex 4 If $u = x^y$ then find $\frac{\partial^3 u}{\partial x \partial y \partial x}$ and show that $\frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial^3 u}{\partial x^2 \cdot \partial y}$

Sol^n:- Given ~~let~~ $u = x^y \Rightarrow \log u = \log(x^y)$.

$$\log u = y \log x$$

Diff. both sides partially w.r.t. x.

$$\frac{1}{u} \cdot \frac{\partial u}{\partial x} = y \cdot \frac{1}{x}.$$

$$\frac{\partial u}{\partial x} = u \cdot \frac{y}{x} \quad -(1)$$

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Hence

Similarly diff. both sides partially wrt y.

$$\frac{1}{u}\frac{\partial u}{\partial y} = \log x \Rightarrow \frac{\partial u}{\partial y} = u \cdot \log x$$

From (1)

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{1}{x}\left[u \cdot 1 + y \cdot \frac{\partial u}{\partial y}\right]$$

$$= \frac{u}{x} + \frac{1}{x}\left[y \cdot u \cdot \log x\right]$$

$$= \frac{u}{x} + \frac{uy \cdot \log x}{x}$$

$$\frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{-u}{x^2} + \frac{1}{x}\frac{\partial u}{\partial x} + y\left[\frac{x \cdot \left(\frac{\partial u}{\partial x} \cdot \log x + \frac{u}{x}\right) - u \log x}{x^2}\right]$$

$$= \frac{-u}{x^2} + \frac{1}{x}\frac{\partial u}{\partial x} + \frac{uy}{x^2} + \frac{y \log x}{x}\frac{\partial u}{\partial x} - uy\frac{\log x}{x^2}$$

$$= \frac{-u}{x^2} + \frac{uy}{x^2} + \frac{uy}{x^2} + \quad - - - - \text{ same } - - -$$

$$= \frac{-u}{x^2} + \frac{2uy}{x^2} + \frac{uy^2 \log x}{x^2} - \frac{uy \log x}{x^2}$$

Now $\frac{\partial u}{\partial y} = u \log x$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{u}{x} + \log x \cdot \frac{\partial u}{\partial x} = \frac{u}{x} + \log x \cdot \frac{uy}{x}$$

Similarly $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{-u}{x^2} + \frac{2uy}{x^2} + \frac{uy^2 \log x}{x^2} - \frac{uy \log x}{x^2}$.

Hence proved

(f). If $x = r\cos\theta$ $y = r\sin\theta$

Then prove that i) $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$ ii) $\frac{1}{r}\frac{\partial x}{\partial \theta} = r\frac{\partial \theta}{\partial x}$

Now. $x = r\cos\theta$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

Also $y = r\sin\theta$

$\Rightarrow x^2 + y^2 = r^2\cos^2\theta + r^2\sin\theta$.

$\Rightarrow x^2 + y^2 = r^2$

Diff both sides partially wrt. x.

$$2x + 0 = 2r\frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} \cdot 2r = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta$$

Clearly $\boxed{\frac{\partial x}{\partial r} = \frac{\partial r}{\partial x} = \cos\theta}$

Now. $\frac{y}{x} = \tan\theta$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right)$$

$$= \frac{1}{x^2+y^2}(-y) = \frac{1}{r^2}(-r\sin\theta) = \frac{1}{r^2}\left(\frac{\partial x}{\partial \theta}\right)$$

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Hence $\frac{\partial\theta}{\partial x} = \frac{1}{r^2}\left(\frac{\partial x}{\partial\theta}\right)$

$$\Rightarrow \boxed{\frac{1}{r}\frac{\partial x}{\partial\theta} = r\cdot\frac{\partial\theta}{\partial x}}$$

Practice Questions :-

1) Show that the function $f(x,y) = \begin{cases} \frac{x^3+2y^3}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

possesses partial derivatives of first order at $(0,0)$

2) If $u = e^{x^2+y^2}$, prove that $x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = 2u\log u$.

3) If $z = \log(x^2+xy+y^2)$, prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2$

4) If $u = \log(x^3+y^3+z^3-3xyz)$

show that

1) $\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)^2 = \frac{9}{(x+y+z)^2}$

2) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

If $V = r^m$, where $r^2 = x^2+y^2+z^2$

show that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = m(m+1)r^{m-2}$.